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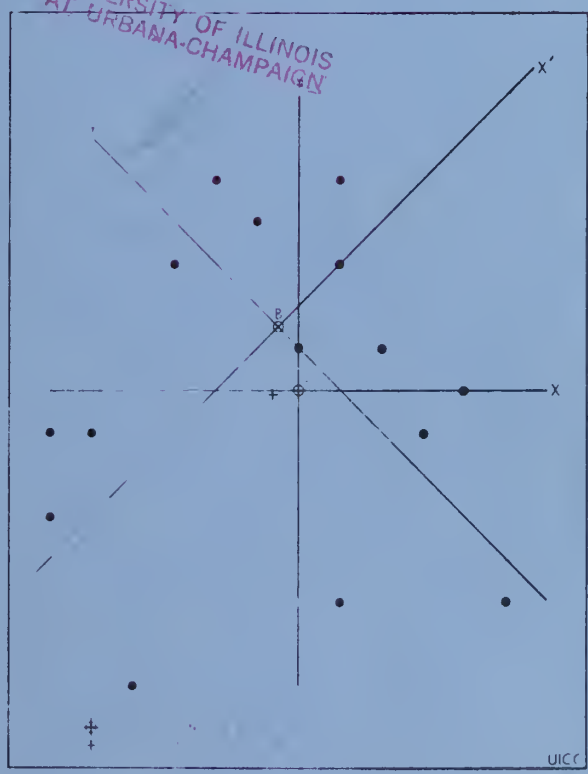
# METHODS AND MEASURES OF CENTROGRAPHY

A Critical Survey of Geographic Applications

SIIM SOOT

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# METHODS AND MEASURES OF CENTROGRAPHY

## A Critical Survey of Geographic Applications

Siim Sööt\*

### ABSTRACT

Centrographic measures have been utilized for decades to draw generalizations about areal distributions. They have proven useful in temporal and comparative analyses in discerning trends and contrasting population distributions. Misconceptions relating to the mathematical derivations, and analytic and descriptive properties of many of the basically simply centrographic techniques which have arisen are explicated in this paper. This is done in a framework of a geographic literature review of three measures of central tendency (mean center, median center and point of minimum aggregate travel) and four measures of dispersion (standard deviational ellipse, bicircular quartic and sectogram). The merits of each measure are specified and contrasted with those of similar measures.

### INTRODUCTION

Centrographic measures have been utilized for analytic and descriptive purposes in a wide range of studies. For example, the center of gravity has frequently been employed to illustrate longitudinal trends in areal distributions (Waterman 1969; Stephenson 1972; Larson and Sööt 1973a; Larson 1973b), while Koch (1942) and Murphy and Spittal (1945) have used it to compare centers of functionally related variables. Also many have used the standard radius or a modification thereof to summarize the dispersion about a point (Yuill 1971; Buttimer 1972; Caprio 1969; Stephenson 1972). Due to their basic simplicity, these measures have a high degree of appeal, but they are analytically limited. Moreover, as generalizing methods they reduce distributions to the most basic characteristics, while overlooking potentially important features.

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The fundamental principles underpinning centrographic measures are elementary, but their applications are marked with misuse and general misinterpretation. Thus, it is necessary to promote proper use of these measures by bringing attention to procedural errors. This paper, then, has a two-fold purpose: (1) to review the centrographic literature with an emphasis on the correction of misconceptions, and (2) to evaluate several centrographic measures.

### MEASURES OF CENTROGRAPHY

Centrographic measures can be used to describe two features of discrete distributions: (1) central tendency, and (2) dispersion.<sup>1</sup> The former is a point which is found by applying at least one averaging criterion. Since there exist several averaging criteria, there are several such points. The most common are the center of gravity (mean center), median center, and the point of minimum aggregate travel. The latter feature, dispersion, depicts the degree of scatter from a point of central tendency. In the two dimensional case, there are numerous techniques and figures which demonstrate the scatter of points: the standard radius, the standard deviational ellipse, the standard deviational bicircular quartic and several versions of the sectogram.

### MEASURES OF CENTRAL TENDENCY

Three univariate measures of central tendency are commonly recognized: mean, median, and mode. The univariate mean has an easily computed and interpreted bivariate counterpart, the center of gravity. The bivariate

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<sup>1</sup> It is the belief of the author that centrographic measures also provide indirect information about pattern, but cannot be classified as useful measures for evaluating the same.



mode is also easily interpreted; it signifies the greatest concentration at one point or within a given area. The univariate median, however, does not have a two dimensional equivalent possessing analogous attributes. The median, as illustrated by Porter (1963), is the point of minimum aggregate travel (PMAT) in the univariate case. But the bivariate PMAT and the (orthogonal) median points are not the same. Each of these bivariate measures will be discussed in greater detail below.

Mean Center. The early prominence of the mean center (center of gravity) is due partially to the fallacious assumption that it represented the PMAT.<sup>2</sup> This confusion was cleared up by Eells (1930). The use of this averaging measure has continued, however, based predominantly on its sensitivity to the location of every point. That is, a shift in any data point will induce at least a minor shift in the mean center. Yet as Hayford (1902, p. 50) points out, there is some objection to the undue weight ascribed to distant points, since by increasing or decreasing in number, they have more effect on the mean center location than close-in points. Still, the measure is attractive since it is computationally simple:

$$(\bar{x}, \bar{y}) \quad \text{where} \quad \bar{x} = \frac{\sum wx}{\sum w} \quad , \quad \bar{y} = \frac{\sum wy}{\sum w} \quad (1)$$

and  $w$  = weight at each point

$x$  = value on an arbitrarily chosen  $x$ -axis

$y$  = value of an axis orthogonal to the  $x$ -axis.

The center of gravity is particularly valuable when used in a comparative analysis with other centers. These other centers may represent either the same variable for a different point in time (Figure 1) or a temporally invariate set of variables (Figure 2). For time variant data sets, the

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<sup>2</sup> The U.S. Bureau of the Census (1921, p. 32) equated the center of population with the point of minimum aggregate travel.



Figure 1. Illinois Centers of Population, 1830-1970.

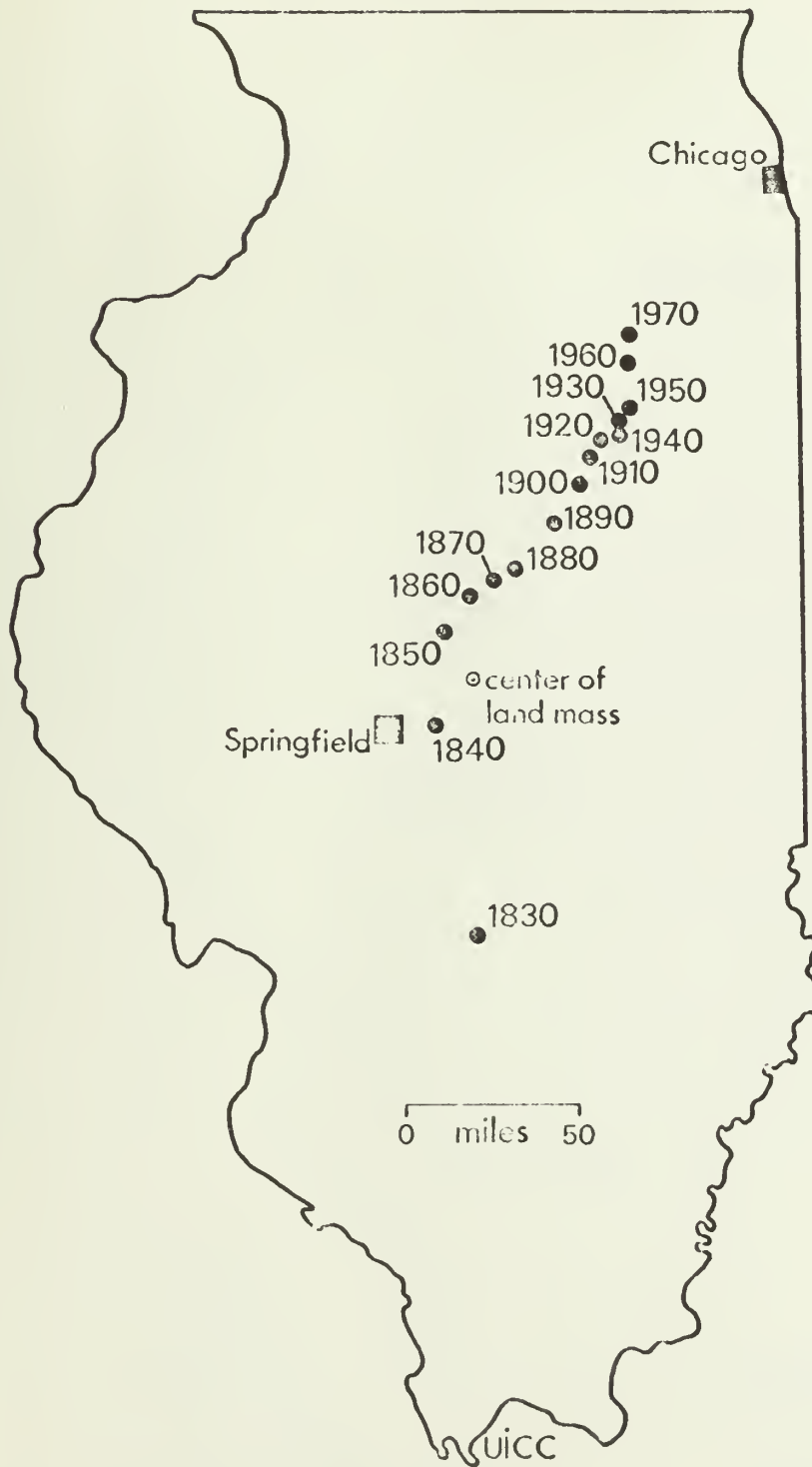
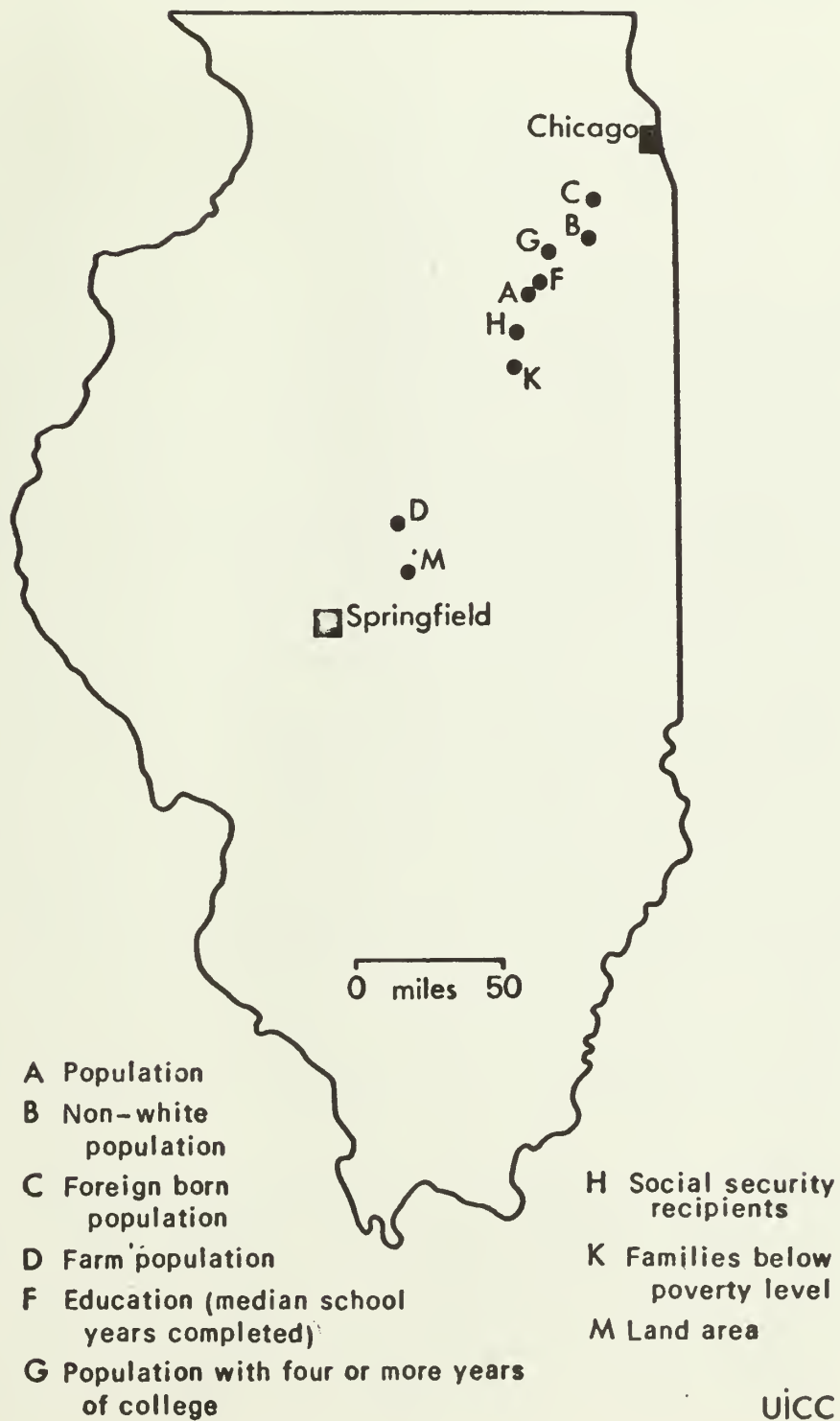




Figure 2. Nine Illinois Mean Centers, 1970.





migration of the centers can be utilized to interpret the sum effect of factors influencing the distribution.<sup>3</sup> Centers of temporally invariate data sets can demonstrate regional imbalances in the respective variables.<sup>4</sup> In each case, one point generalizes an entire weighted distribution. Figure 3 depicts the population distribution summarized by the 1970 mean center on Figure 1.

In analyzing the movement of mean centers, the direction and rate of migration are particularly important. Janelle (1971) in examining residential, public, commercial, and industrial land use surfaces, goes one step further. He suggests the concepts of velocity, acceleration, and momentum to describe the shifts in the mean center and thereby the data they represent. The velocity is the rate of shift of the mean center in a given direction, the acceleration is the rate of increase in velocity, and the momentum is a product of the mass and velocity. Of the three, velocity and momentum are the most practical.

Velocity is useful in that it indicates the distance and direction moved per unit time period. The velocity vectors, as shown for Illinois in Figure 4, designate the utility of this statistic. Note that the post-1940 vectors are easily distinguishable from the 1900-1930 vectors.

The advantage of momentum is that it compensates for the deceleration occurring with population increases. But since momentum is a product of

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<sup>3</sup> Figure 1 captures exceptionally well the almost relentless movement of the center toward Chicago. The only reversal, during the 1930-1940 decade, was precipitated by a period of economic insecurity, as the migration from the farm to the city reversed directions. This decrease in urban migration was significant enough to cause the only southward movement of the center over a 140 year period. The population centers also suggest why Springfield was selected the State Capital, as it was the urban place closest to the 1840 center.

<sup>4</sup> For example, Figure 2 illustrates the degree to which the distribution of foreign born population (C) is biased toward the northeast.



Figure 3. 1970 Illinois Population by Counties

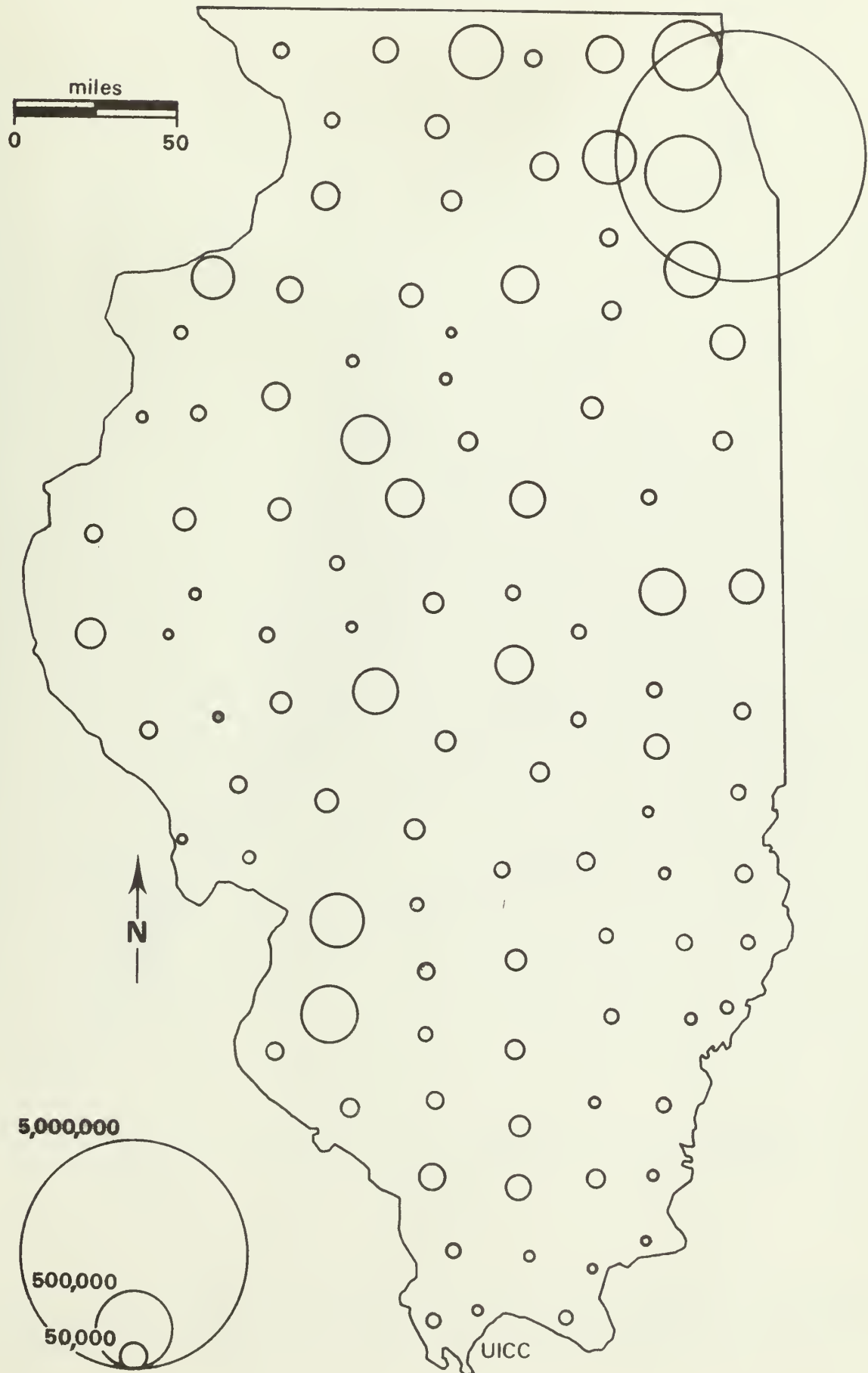
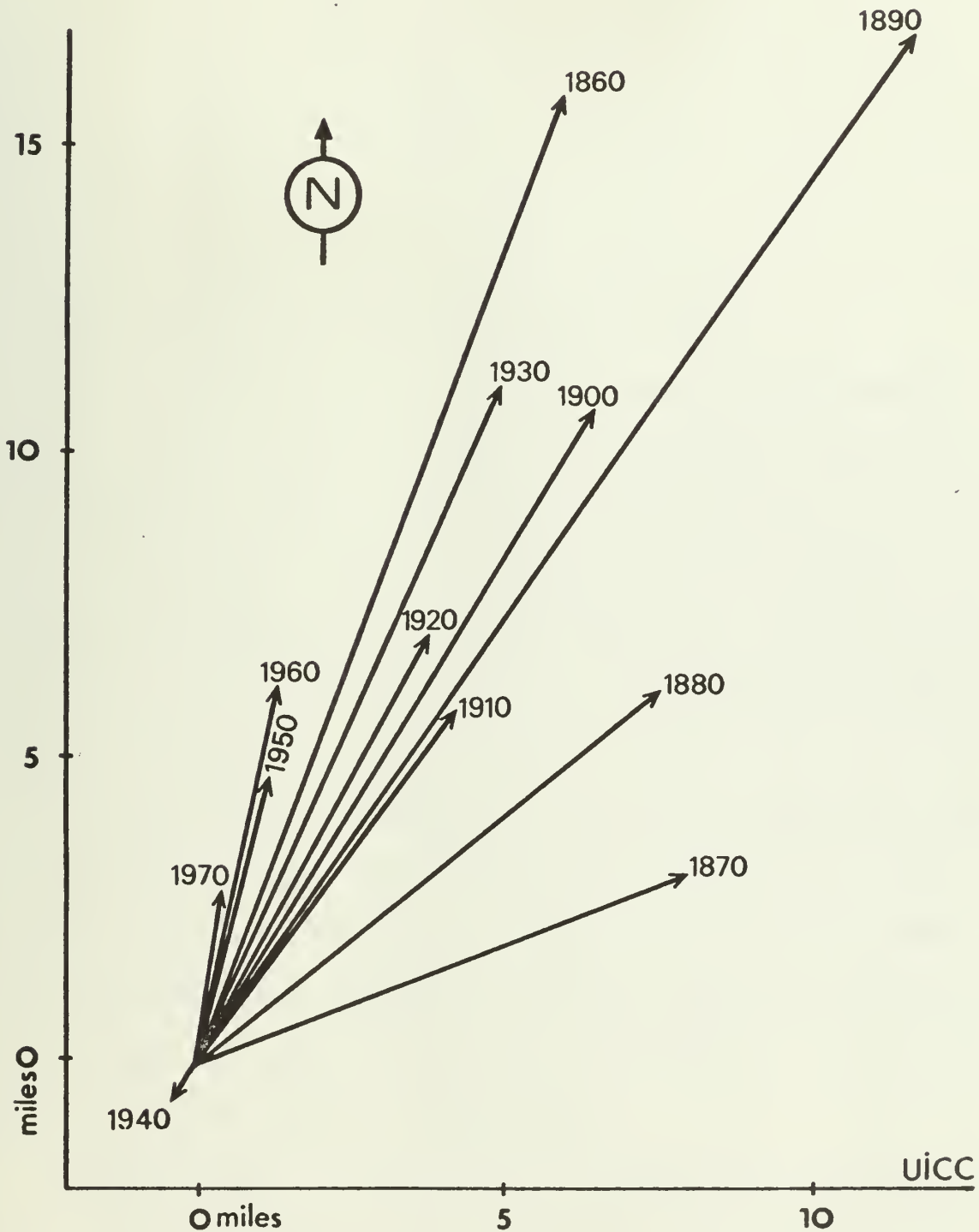




Figure 4. Velocity Vectors of Population Centers, Illinois 1850-1970.



The vector origins are placed at a common point to illustrate the pattern of movement over the last 110 years. Two irregular trends can be observed: the movement is approaching the north-south axis, and the speed is decreasing.



mass and velocity, it can increase even if speed decreases. It, therefore, permits comparisons between the total impacts of population redistribution in the early growth of Illinois and the most recent decades. It is essentially a means of "standardizing" the data for comparative purposes. Notice that the momentum increased from 1830 to 1880 while the speed was decreasing (Table 1). In short, the momentum shows that the greatest thrust of population redistribution toward Chicago took place during the prosperous 1920's while the 1880-1890 decade was second, followed by the rural-urban migration of the 1950's.

The acceleration of the mean centers can also be computed but it is felt that such an exercise in this problem would add little to what is already apparent in Figure 4 and it would only over-quantify the population redistribution data. A cursory observation of Figure 4 reveals that acceleration (deceleration) was great before the turn of the century and relatively small since 1930. This is not unexpected since centers of growing populations are characterized by deceleration.

Although Janelle's methods are mathematically sound they have not received widespread use. Perhaps the primary reason is the lack of reliable temporal data in the consistent geographic units to examine these trends. Aside from population figures how many data sets span one hundred years? Also the measures such as acceleration are frequently too abstract to be useful in a wide variety of circumstances.

Median Center. The median center suffers from a major deficiency: it is not a unique point. This deficiency arises from the method of determination. The point is defined as the intersection of two orthogonal axes, each of which divides the distribution into equal halves.<sup>5</sup> As the

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<sup>5</sup> The measure is, therefore, also known as the orthogonal median point.



Table 1. Movement of the Illinois Population Center, 1830-1970.

Mass, Speed, Acceleration, and Momentum

| Decade    | Mass <sup>a</sup> | Distance or speed <sup>b</sup> | Acceleration <sup>c</sup> | Momentum <sup>d</sup> |
|-----------|-------------------|--------------------------------|---------------------------|-----------------------|
| 1830-1840 | 316               | 56.2                           |                           | 1776                  |
| 1840-1850 | 663               | 33.0                           | -23.0                     | 2187                  |
| 1850-1860 | 1281              | 16.6                           | - 6.4                     | 2120                  |
| 1860-1870 | 2125              | 8.4                            | - 8.2                     | 1787                  |
| 1870-1880 | 2808              | 9.5                            | 1.1                       | 2675                  |
| 1880-1890 | 3452              | 20.3                           | 10.8                      | 7015                  |
| 1890-1900 | 4324              | 12.4                           | - 7.9                     | 5361                  |
| 1900-1910 | 5230              | 7.1                            | - 5.3                     | 3696                  |
| 1910-1920 | 6062              | 7.7                            | 0.6                       | 4695                  |
| 1920-1930 | 7058              | 11.9                           | 4.2                       | 8381                  |
| 1930-1940 | 7763              | 0.9                            | -10.0                     | 699                   |
| 1940-1950 | 8304              | 4.7                            | 3.8                       | 3873                  |
| 1950-1960 | 9397              | 6.2                            | 1.5                       | 5805                  |
| 1960-1970 | 10598             | 2.8                            | - 3.4                     | 2919                  |

<sup>a</sup>Average population in thousands.<sup>b</sup>Distance in miles per decade and speed are the same.<sup>c</sup>Per decade change in speed.<sup>d</sup>Mass times distance expressed in tens of thousands of population miles.



orthogonal axis system is rotated, a shift is necessary to maintain the median quality of the axes (Figure 5). Since generally no one axis orientation can be justified over another, there is no rationale for selecting one median center. However, the median center is less influenced by extreme data points than the mean center (Prunty, 1951, p. 202) such that the points may be moved anywhere within their quadrant (Figure 5) without affecting the median center. The median center also is much more easily determined than the mean center, but it is not the point of minimum aggregate travel, as is true in the univariate case (Porter, 1963).

A city with a grid transportation network, however, represents an important exception. If all the movement in an area is restricted to streets parallel to one of two orthogonal axes (grid pattern), then these axes can define a unique center which the author contends is also the PMAT. In the bivariate case if we collapse the two dimensional distribution to points on two perpendicular axes, then the median can be computed individually on each axis (Figure 6). One axis can then be moved, without disturbing orthogonality, so that the two median points coincide. This point then is the PMAT, since all travel can be interpreted as being confined to the x and y axes. By the procedure outlined above, travel on these axes has been minimized. Thus, in any system in which movement is confined to a grid network, the PMAT is also the orthogonal median point. Although this is a special case, several cities and some agricultural areas approximate the perfect grid system. The spacing of the streets, of course, is not important but parallelism and orthogonality must be maintained.

Point of Minimum Aggregate Travel (PMAT). The PMAT has significant practical value as it designates the point to which the total travel by all in the study area is minimized. For example, in locating a state



Figure 5. The Median Centers and Axes Rotation.

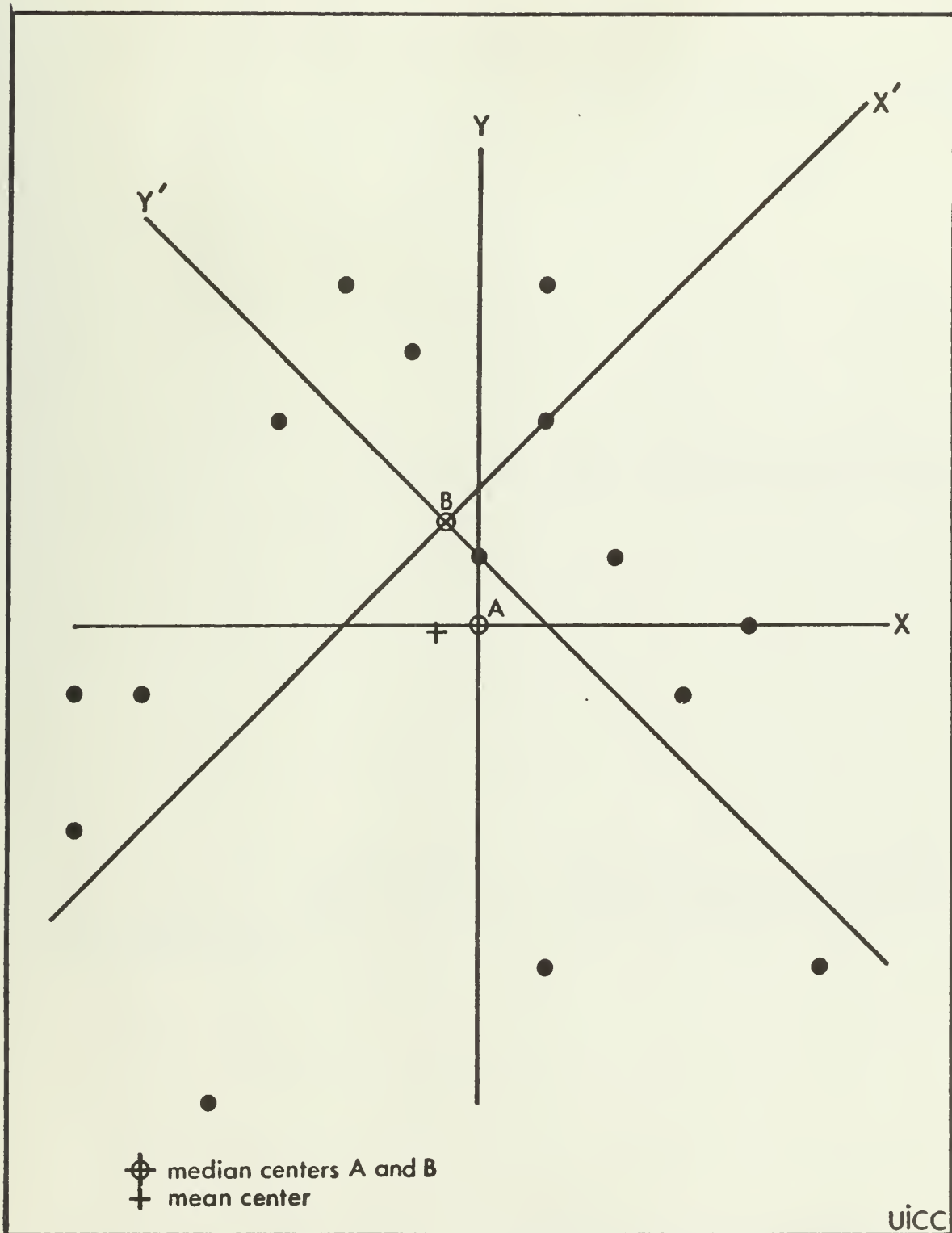
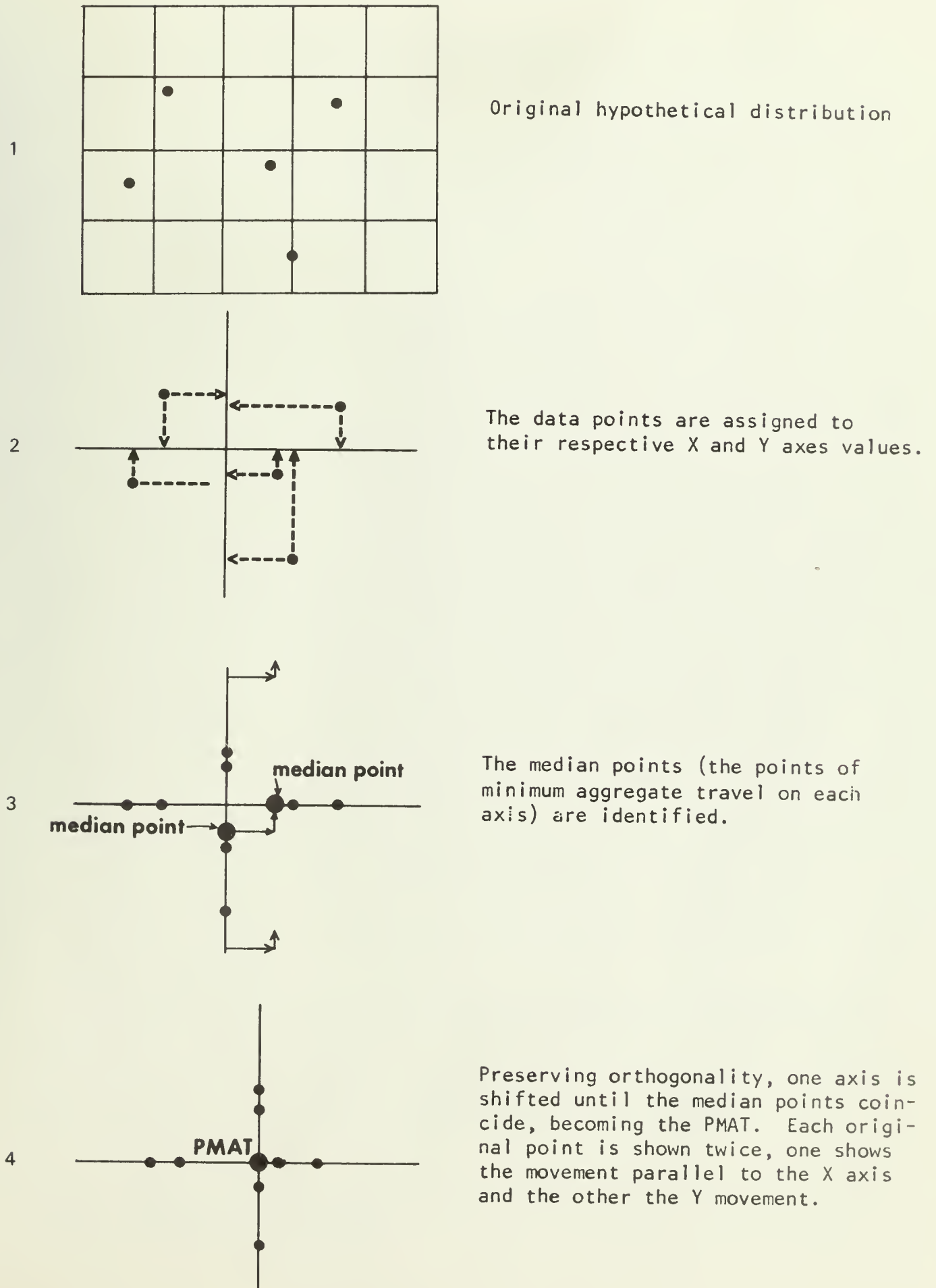




Figure 6. The Median Center and the Point of Minimum Aggregate Travel.





capital, if the objective is to minimize total travel time, it should be located at that point.

The precise determination of this point had perplexed mathematicians for centuries, but it is now accepted as having no mathematical solution.<sup>6</sup> To date, the best methods remain iteration techniques. Seymour (1965) suggests placing a uniform lattice of points (intersections of a regularly spaced grid system) over the study area and identifying the point with the smallest aggregate travel (Figure 7). This point becomes the center of a finer gride of points, extending to a perimeter delimited by the closest points in the original lattice. Successive reapplications of this technique produce a close approximation of the PMAT.

Despite the lack of a mathematical formula, computer programs can easily provide iterative solutions, and rather accurate estimates can be computed. In practical applications, only approximations are necessary since the solution (location) is often not suited for the point in question. For example, if a supermarket can estimate the scope of its trade area and frequency of business, the PMAT can be determined for a unit time period. That point may be located in a park, cemetery, or any of a list of areas not zoned for commercial activity or merely not desirable for large scale retail trade. Moreover, judgment is necessary in estimating future shifts in the trade area. All this negates the necessity of precise computations.

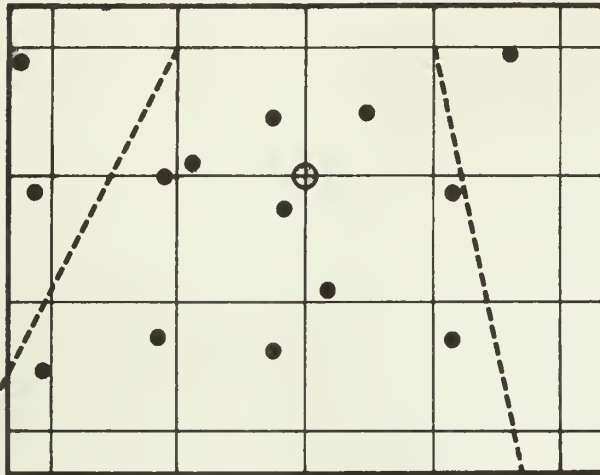
The State of Illinois may be utilized to illustrate the location of the PMAT and its practical implications. With nearly two-thirds of the state's population, the Chicago area has already been demonstrated to

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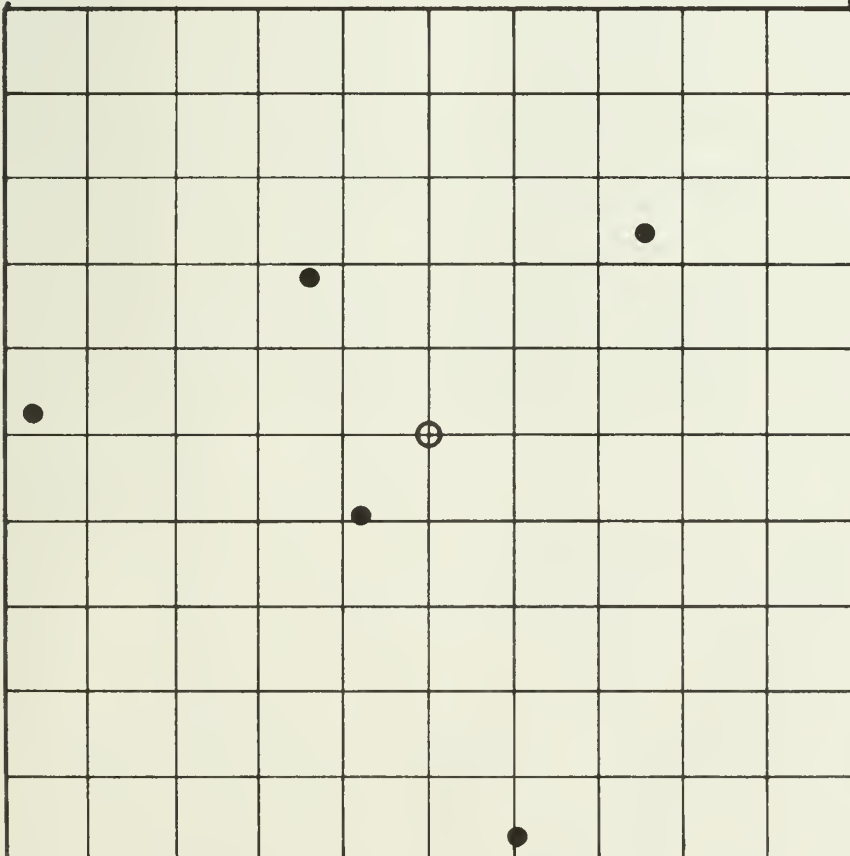
<sup>6</sup> Porter (1963) proposed a geometric solution, but Court (1964) showed Porter's method to be spurious.



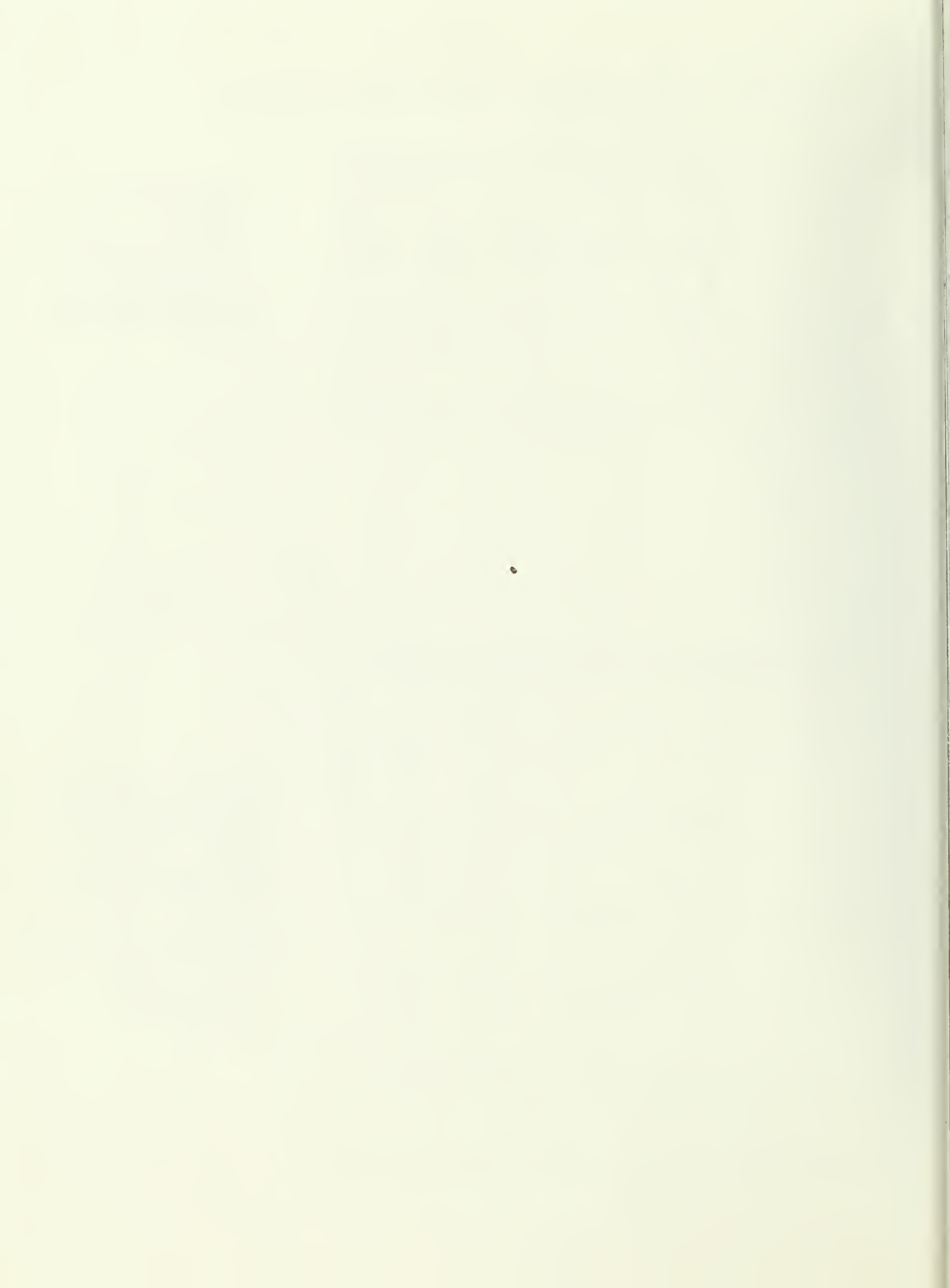
Figure 7. One Iterative Step in the Derivation of the PMAT.



A hypothetical distribution of potential customers. A regular grid is superimposed.  $\theta$  is found to be the grid intersect with the lowest aggregate travel.



A finer regular grid is used to calculate new aggregate travel values, and a new minimum is identified (not marked on this figure). The grid surrounding that point can again be magnified and the procedure repeated until the desired accuracy is achieved.



have a powerful influence on the mean center. This influence is even stronger in the case of the PMAT; it is located within the City of Chicago. Is it then advisable to locate all state-wide services in Chicago? Not necessarily, since this statistic is relatively insensitive to the locations of the most distant points. The fact that Cairo is four hundred miles to the south is of little consequence. In this context a point further south would probably be more suitable. The mean center is one such point.

The three bivariate measures of central tendency represent distinctive features of the distributions as well as unique computational procedures. The mean center minimizes the sum of the squares of the deviations to each point, the PMAT minimizes the sum of the absolute deviations and since the median center is not unique it is not a point of minimization (except in circumstances when the movement is confined to a grid network). Computationally, the median center is the most readily determined since it is the only measure which does not require conversion to cartesian coordinates. On the other hand, the PMAT is the most difficult to compute, but it is the point with the most practical and theoretical value. Since these points have different characteristics, they generally are not found at the same location--the bivariate normal distribution being the major exception.

#### MEASURES OF DISPERSION

A measure of central tendency is commonly, though not always, the reference point for measures of dispersion. Dispersion measures are useful descriptive devices for expressing the degree of concentration or scatter and in some cases for specifying directional orientations of the distribution.



The use of these measures involves two basic decisions: (1) which point of central tendency should be selected as a reference, and (2) which measure of dispersion should be utilized. Neither is easily answered, but a few fundamental principles can be deduced. The critical factor in each case appears to be the desired mix of analysis and description.<sup>7</sup> If description is the principal objective then a multitude of measures of central tendency and dispersion may be suitable, depending upon the problem. On the other hand, for strictly analytic purposes the standard radius based on the mean center should be used. In limited cases the standard deviational ellipse may also be considered an analytic tool.

Although choosing the appropriate reference point for the dispersion measure is often an elementary process, there is no universal agreement regarding the selection of the appropriate point. Hurst and Seller (1969, p. 184), for example, disagree with Blount's (1964) selection of the "center of gravity over which to place the normal curve" (circular normal probability surface).<sup>8</sup> Since Blount is examining the distribution of shoppers, Hurst contends that the shopping center should be the point from which to measure dispersion. If the purpose is description then the shopping center would seem more appropriate. But Blount's test for circular normality requires the mean center, although the reason for testing for circular normality of shoppers is not clarified. Hurst also tests for circular normality, but, inappropriately, from the shopping site instead of from the center of gravity. If the shopping site deviates significantly

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<sup>7</sup> An analytic technique is herein considered to be a method which permits the testing of hypotheses. Other techniques which merely convey informational characteristics about a distribution are descriptive.

<sup>8</sup> The circular normal probability surface is defined in the section on standard radius.



from the mean center, one may more quickly reject circular normality. However, if the two points coincide, one would have to test for circular normality by determining the distribution by distance zones.

The selection of the reference for the dispersion measure is then mainly dependent upon the objective of the study. Tests for bivariate normality require the mean center. Descriptive studies may employ the median center, the point of minimum aggregate travel or any data point or non-data point location within the study area.

There is also disagreement regarding the utility and interpretation of measures of dispersion. Four methods, (1) standard radius, (2) standard deviational ellipse, (3) standard deviational bicircular quartic, and (4) sectogram, are discussed and evaluated.

Standard Radius (SR). The two dimensional counterpart of the univariate standard deviation is the standard radius. As such it is a widely used analytic and descriptive measure of scatter.<sup>9</sup> It may be used: (1) to test for circular normality; (2) to express degrees of dispersion for a circular normal distribution; and (3) to express dispersion for non-normal distributions. For example, Shachar (1967) and Waterman (1969) used the SR to depict scatters of central place functions in Tel Aviv and to describe the population distribution of Ireland from 1841 to 1966, respectively, although these distributions deviated significantly from the bivariate normal. Also, Stephenson (1972) uses the standard radius to

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<sup>9</sup> The standard radius is also referred to as the standard distance, and by Yuill (1971) and Caprio (1969) as Bachi's standard distance. Although Bachi is credited with reviving this measure it was defined by Furfey in 1927 and probably by statisticians before him. Hultquist et. al. (1971) and Lee (1966) call the standard deviation along an axis the standard distance. This can be a useful distinction when distances are referenced to a line, but it is also indicative of the lack of agreement in centographic nomenclature.



illustrate crime patterns in Phoenix. Each of these three applications are legitimate descriptive uses, since no analytic inferences are implied.

If a distribution is normal, then incremental standard radius bands encompass easily determined percentages of data points (similar to the manner in which the univariate standard deviation is used). The percentage of points in each band is used as a test for circular normality. Burlington and May (1953, p. 97) demonstrate that 63.2% of the data points lie within one standard radius. Neft and Warntz (1960) provide a complete probability table of 0.01 increments of the SR. The probabilities approximate, but do not equal, those of the univariate normal distribution.

Mathematically, the standard radius is expressed by the relationship

$$SR = \sqrt{\sigma_x^2 + \sigma_y^2}, \quad (2)$$

where  $\sigma_x$  = the standard deviation along the x-axis, and  $\sigma_y$  = the standard deviation along the y-axis. It is, in essence, the sum of two orthogonal vectors representing the standard deviations along their respective axes (Figure 8). In the case illustrated, the distribution is symmetric ( $x=y$ ) and the standard radius (OC) is the sum of vectors OB and BC (or OA). The SR is also defined as the square root of the mean of the squared radial distances from the origin:

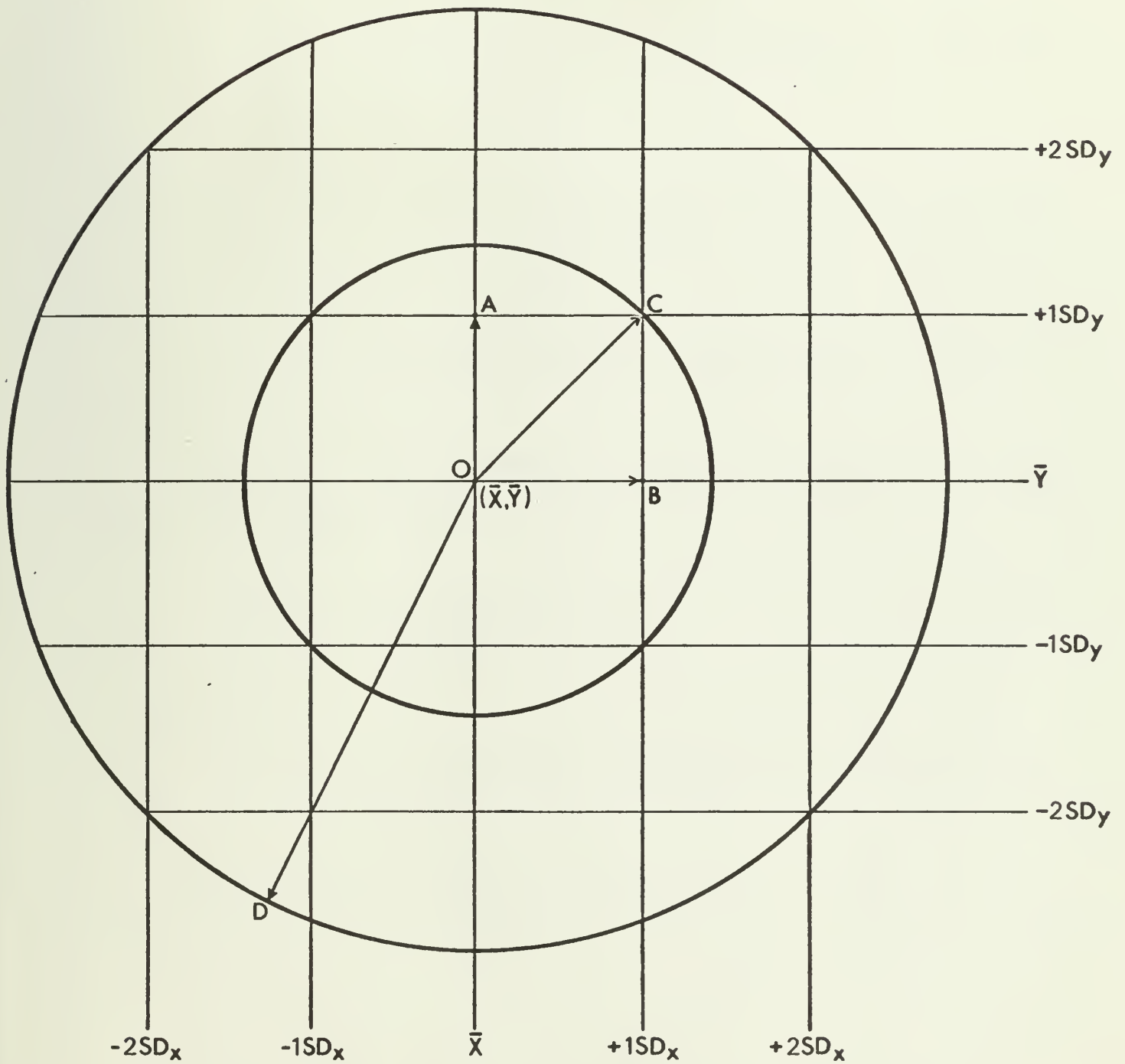
$$SR = \sqrt{\frac{\sum d_i^2}{n}}, \quad (3)$$

where  $d_i$  = the radial distance from the origin to any data point  $i$ , and  $n$  = the number of data points.

Like the center of gravity, the standard radius is insensitive to the orientation of an orthogonal axis system (see also Waterman 1969, p. 54).



Figure 8. Standard Radius and the Bivariate (Circular) Normal Distribution



- $\vec{OB}$  Standard deviation on the X axis
- $\vec{OA}$  Standard deviation on the Y axis
- $\vec{OC}$  Standard radius
- $\vec{OD}$  Two standard radii



This property is illustrated in Figure 9. Regardless of the axis orientation, defined by AOB, COD or FOG the SR remains constant ( $OR = OD$ ).

Although numerous tables for standard radii are available, the probability of a point falling within a standard radius can also be derived by integral calculus. A series of procedural steps outlined below summarize the mathematical logic. The sequence proceeds from the most general equation to a specific form, without deserting generality. The bivariate normal surface will be translated to the special case in which the means of  $X$  and  $Y$  equal zero, their standard deviations are unity, and there is no correlation between  $x$  and  $y$  (i.e.,  $\mu_x = \mu_y = 0$ ,  $\sigma_x = \sigma_y = 1$ , and  $\rho_{xy} = 0$ ). This describes the circular normal distribution with unit variance centered on the origin.

The general form for the bivariate normal (Gaussian) surface is (Burrington and May 1953, p. 97):

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-Q(x,y)}, \quad (4)$$

where

$$Q(x,y) = \frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right]. \quad (5)$$

Since we are mainly concerned with the circular normal distribution, the equivalence of the two standard deviations ( $\sigma_x = \sigma_y = \sigma$ ) can be utilized and

$$f(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_x)^2 - 2\rho(x-\mu_x)(y-\mu_y) + (y-\mu_y)^2}{\sigma^2} \right]}. \quad (6)$$



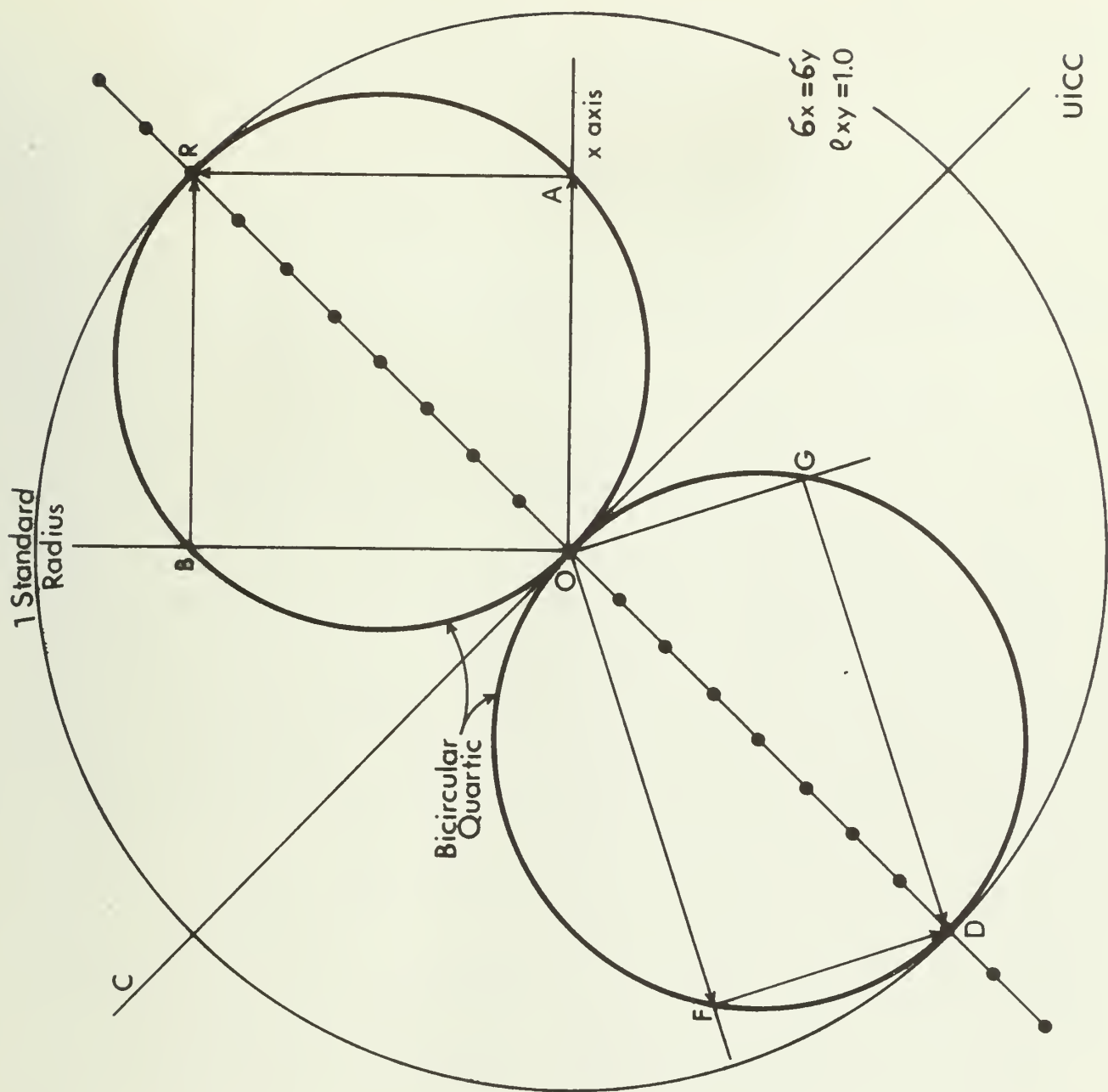


Figure 9. Standard Radius and the Bicircular Quartic for a Distribution of Points along a Straight Line\*

OA, OB, OG, and OF are standard deviations on their respective axes.

OR and OD are standard radii.

\* Not all of the points are shown; some extend beyond those shown outside the standard radius.



Further, the independence of the two axes ( $\rho_{xy} = 0$ ) can be introduced since the two variables  $(x,y)$  are normally distributed:<sup>10</sup>

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-1/2 \left[ \frac{(x-\mu_x)^2 + (y-\mu_y)^2}{\sigma^2} \right]}. \quad (7)$$

The equation may be further simplified by translating the system to the center of gravity ( $\mu_x = \mu_y = 0$ ):

$$f(x,y) = \frac{1}{2\pi\sigma} e^{-1/2 \left( \frac{x^2 + y^2}{\sigma^2} \right)}. \quad (8)$$

Lastly, the variance is reduced to unity ( $\sigma = 1$ ) and

$$f(x,y) = \frac{1}{2\pi} e^{-1/2(x^2 + y^2)}. \quad (9)$$

The probability that a point falls within a circle,  $C$ , is given by:

$$p(c) = \iint_C f(x,y) dx dy. \quad (10)$$

By changing to polar coordinates  $(r, \theta)$  and integrating over a distance of one standard radius through one revolution we have:

$$\begin{aligned} p(C) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{SR} e^{-\frac{r^2}{2}} dr d\theta \\ &= 0.632. \end{aligned} \quad (11)$$

This procedure demonstrates that for a circular normal distribution 63.2% of the points lie within one standard radius of the mean center. Similar

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<sup>10</sup>This is not true for all random variables  $(x,y)$ , but applied to normally distributed variables. See Birnbaum (1962, p. 153) for a formal proof.



integration shows that a circle with the univariate standard deviation<sup>11</sup> as a radius only encompasses approximately 40% of the points (a circle of radius OA in Figure 8). However, when a bivariate normal distribution is not circular, the analytic test of normality requires considerably modified tables. Table 2 lists probabilities for such distributions.<sup>12</sup> It is important to understand that the bivariate normal distribution does not exhibit some of the properties of the univariate normal distribution; therefore, they must be treated as two different entities. If all the points in a bivariate normal distribution are converted to vector distances, the resulting distances are not equivalent to one tail of the normal distribution. A bivariate distribution with a standard radius encompassing 63.2% of the points is normally distributed, whereas a univariate distribution with a standard deviation encompassing 63.2% of the points is not normal. Similarly, in a three dimensional case, only one of the characteristics of a series of spheres--the diameter, the surface area or the volume--can be normally distributed. Thus, the univariate test cannot be employed to determine circular normality of a two dimensional distribution. This point has evaded Blount (1964), Hurst and Sellers (1969), Fitzgerald (1974), and Davis (1974).

The fundamental disadvantage of the SR is that it is primarily a test of circular normality. As a descriptive tool it requires a circular distribution. For instance, the Illinois SR extends beyond the boundaries of the state, and for this reason, it is obviously not a good measure of scatter (Figure 10). A measure of scatter should delimit the area of

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<sup>11</sup> There is, of course, no such thing as a standard deviation in a two dimensional distribution.

<sup>12</sup> Groenewood et al. (1967) provide an extensive table of bivariate normal probabilities.



TABLE 2

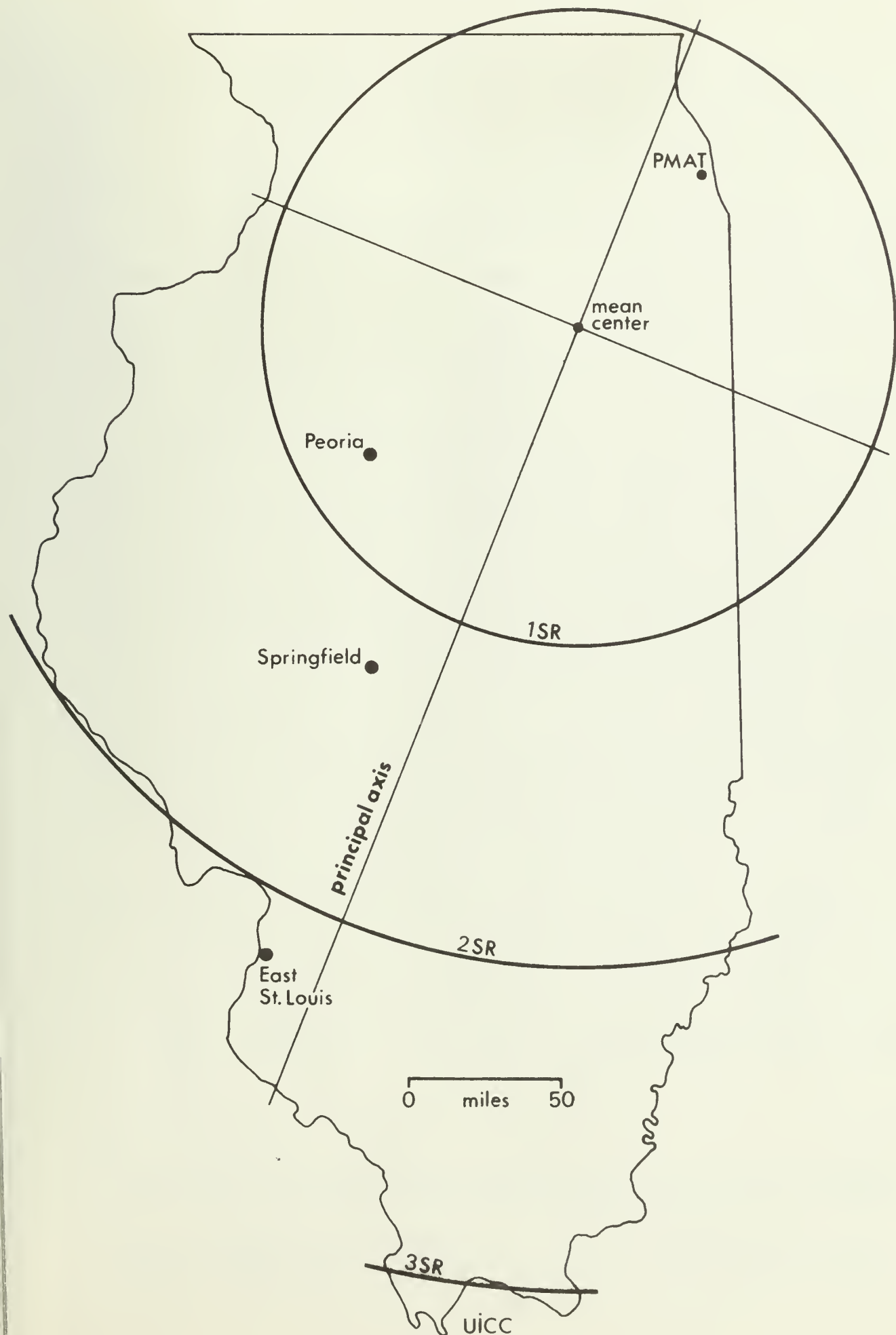
PROBABILITY OF A POINT FALLING WITHIN ONE STANDARD DEVIATION AND ONE STANDARD RADIUS OF THE MEAN CENTER IN A BIVARIATE NORMAL DISTRIBUTION

| Ratio |   | Probability |   |        |
|-------|---|-------------|---|--------|
|       | : | one SD      | : | one SR |
| 0.0   | : | .6826       | : | .6826  |
| 0.1   | : | .6802       | : | .6824  |
| 0.2   | : | .6724       | : | .6818  |
| 0.3   | : | .6568       | : | .6785  |
| 0.4   | : | .6291       | : | .6732  |
| 0.5   | : | .5901       | : | .6634  |
| 0.6   | : | .5461       | : | .6541  |
| 0.7   | : | .5026       | : | .6424  |
| 0.8   | : | .4621       | : | .6362  |
| 0.9   | : | .4258       | : | .6335  |
| 1.0   | : | .3935       | : | .6320  |

Source: Groenewoud (1967).



Figure 10. The 1970 Illinois Mean Center and the Standard Radius.





highest concentration, while not extending beyond the study area; however, such an occurrence is difficult to avoid when the distribution is confined largely to a limited area near the perimeter. In excluding most of Illinois, the circle illustrates the dominance of Chicago in the state's population distribution.

This disadvantage has led to modifications of this measure to improve its descriptive qualities. All these modifications represent departures from the analytic realm, and are inherently descriptive in nature, thus they must be judged on their descriptive merits.<sup>13</sup>

Standard Deviation Ellipse (SDE). Due to the insensitivity of the SR to strong directional trends in distributions, Lee (1966) and Yuill (1971) recommend an ellipse for description. The ellipse is a well understood geometric form which is easily described and readily computed. The SDE is centered on the mean center with the major axis of the ellipse being the principal axis least squares line which minimizes the sum of the squared perpendicular deviations. The minor axis is perpendicular to the major axis at the mean center. The respective standard deviations along these two axes define the standard deviation ellipse (Figure 11).

Lee (1966) further justifies its use by showing that a bivariate normal surface, cut by a plane parallel to the (x,y) plane traces an ellipse,<sup>14</sup> indicating that, in some circumstances, the ellipse can be a test for a bivariate normal distribution. Since the bivariate normal is an uncommon distribution, the ellipse is almost always used as a descriptive tool.

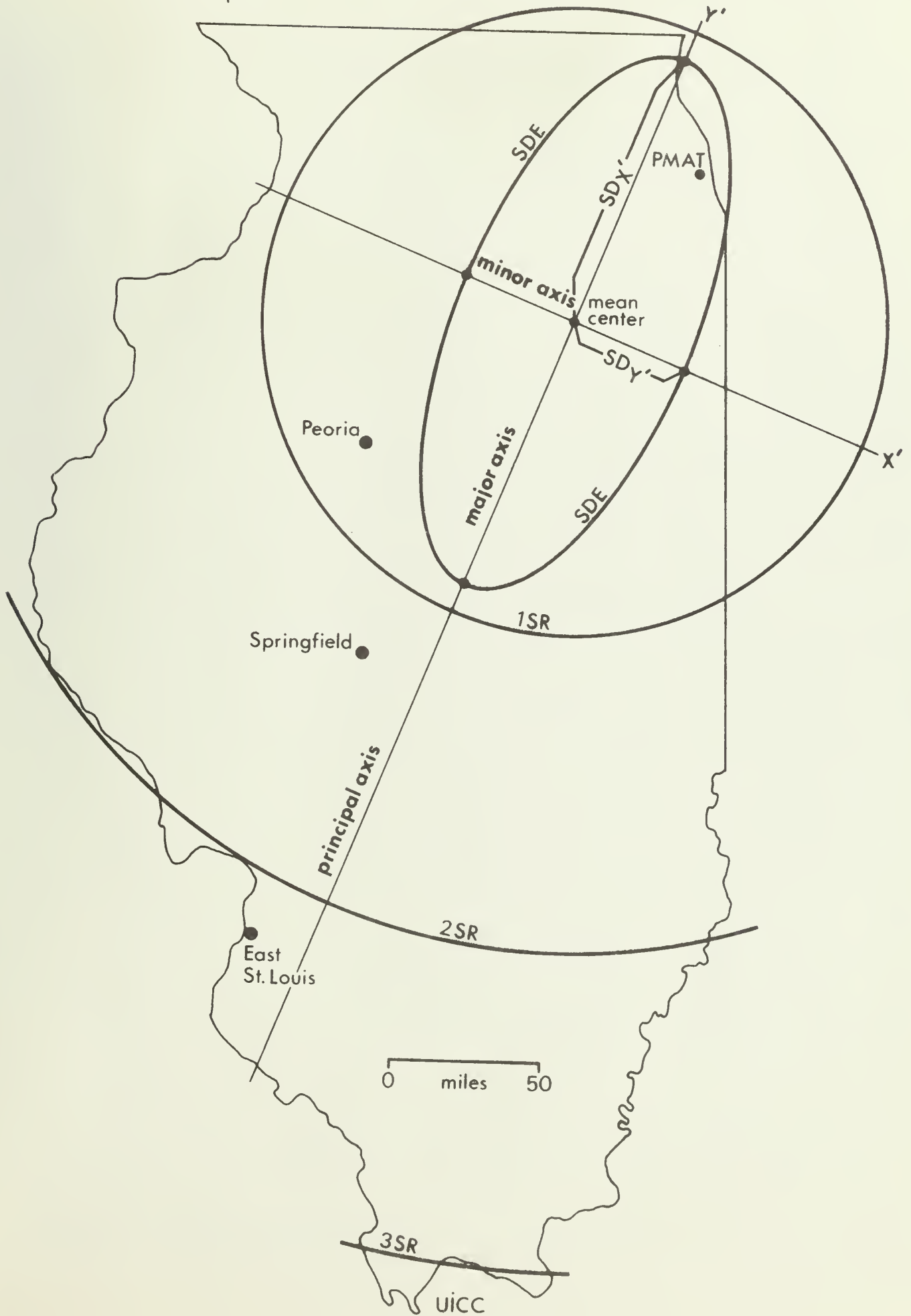
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<sup>13</sup>The descriptive qualities may be determined by how well the measure communicates the distribution given only the measure as an indicator.

<sup>14</sup>When the major and minor axes are twice the respective univariate standard deviations, the ellipse is known as the standard ellipse.



Figure 11. The 1970 Illinois Standard Radius and the Standard Deviation Ellipse.





Yuill (1971, pp. 30-31), although acknowledging some of the methodological shortcomings, also extols the virtues of the ellipse. He presents an intricate trigonometric procedure for determining the nature of the ellipse and specifically the axis orientation. He fails to point out, however, that the axis which minimizes the standard deviation is the principal axis. As such, no trigonometric procedure, although interesting, is essential. His advocacy of the SDE is accepted, but Yuill recommends placing the ellipse on the median center rather than on the mean center. The use of the median center raises the question of orientation. Since the location of the median center varies with the orientation of the reference axes, an ellipse placed on the median center, rather than on the mean center, will have no unique orientation. Although the same orientation as used with the mean center was implied, it was not so expressed nor was a justification for maintaining this orientation provided.

Ellipses have numerous properties--circularity, area, and orientation--by which they can be compared (Hultquist, et al., 1971). Nevertheless, the ellipse has certain shortcomings such as its inability to capture specific directional biases, and its limited analytic utility. Using Table 2, on a more complete version, the ellipse may be used to test for a bivariate normal distribution. This is done by producing successive ellipses with their major and minor axes being fractional multiples of their respective standard deviations. Although the figure is different, the procedure parallels that of the test for circular normality. As a descriptive tool, its value is completely dependent upon the shape of the distribution.<sup>15</sup> If a distribution is (1) multi-modal, (2) uniform, (3) random, or (4) pear-shaped, then an ellipse is not a good descriptive form. When the

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<sup>15</sup>Several examples may be found in Yuill (1971).



distribution approaches an elliptical form, it is a good descriptive figure. Presented alone, however, the ellipse may disguise the shape of the distribution, in which case it may not truly represent the underlying distribution. When the purpose is to indicate the general shift over time, or to compare several variables, then the ellipse proves useful in contrasting the distributions in question.<sup>16</sup>

Perhaps if the SDE were to receive universal adoption, researchers could develop an intuitive feel for what the ellipse is describing. Such an adoption is not warranted due to the limited descriptive capabilities of the ellipse. The weak link, then, is the limitation of the ellipse in describing the point scatter, rather than finding ways to describe the ellipse. Used with this limitation in mind, it still may prove a practical measure for contrasting spatial distribution.

Standard Deviational Bicircular Quartic (BCQ). The basic procedure in delimiting the ellipse is slightly modified to produce the BCQ. Rather than defining the figure with two axes and four points, the BCQ consists of a locus of points (Figure 12). These points are derived by computing the standard deviations of each of two orthogonal axes anchored at the mean center and incrementally rotated by 90° (or rotating one axis 180°).<sup>17</sup> This figure is then designed to be responsive to specific directional biases in the distribution, but as will be evident shortly, it fails to be a significant improvement over the SDE.

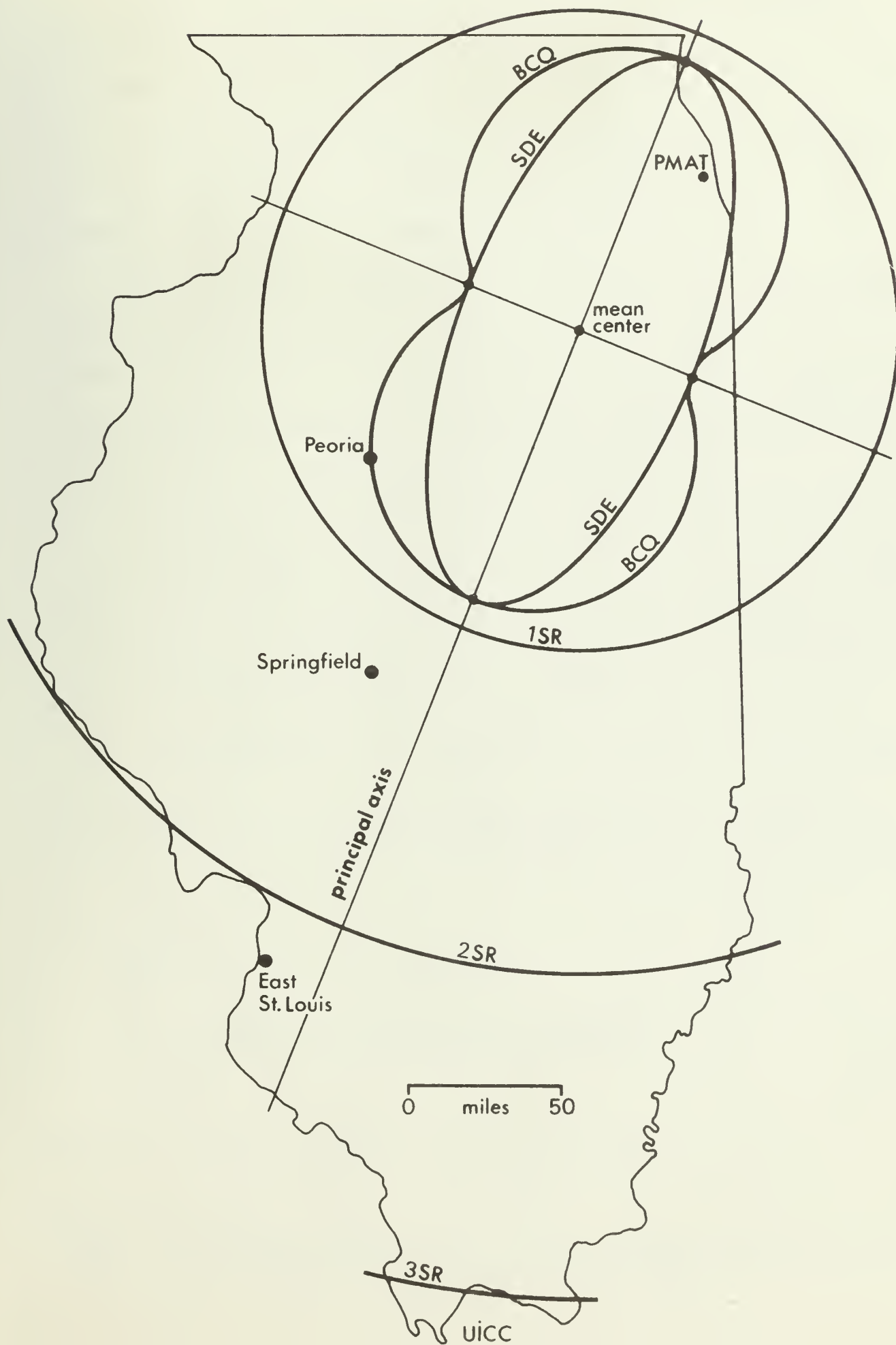
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<sup>16</sup>The ellipse is frequently used in this manner, for example see: Caprio (1969), Buttner (1972), and Hyland (1970).

<sup>17</sup>The figure eight in Figure 9 has been produced by such a procedure. Notice that the standard deviation of the axis perpendicular to the line connecting the data points is zero.



Figure 12. Three Measures of 1970 Illinois Population Dispersion.





Lefever (1926) first recommended this technique but mistakenly called the resulting figure an ellipse. Furfey (1927) corrected this misconception and provided a mathematical argument to demonstrate that the figure is, in fact, a bicircular quartic (a fourth degree equation consisting mainly of two circles). One limiting case for such a geometric form is illustrated in Figure 9; the other limit is a circle.

The mathematical derivation of this figure can best be understood by assigning each point a new set of coordinate values produced by an axes rotation of angle  $\theta$ .<sup>18</sup> By rotating the axes and computing the respective standard deviations the figure is delimited. The transformation of coordinate values is performed with the following equations:

$$x' = y \sin\theta + x \cos\theta \quad (12)$$

$$y' = y \cos\theta + x \sin\theta, \quad (13)$$

where  $(x,y)$  is the original point and  $(x', y')$  is the transformed point.

For the rotated axis system the  $\sigma_{x'}$  becomes:

$$\sigma_{x'} = \sqrt{\frac{\sum (x')^2}{n} - (\bar{x}')^2} = \sqrt{\frac{\sum (x')^2}{n}} \quad (14)$$

The average term,  $(\bar{x}')^2$ , drops out since the system is centered on the center of gravity ( $\bar{x}' = \bar{y}' = 0$ ). This center is a unique point, independent of axis rotations.

Substituting (12) into (14) we have:

$$r = \sqrt{\frac{\sum y^2 \sin^2\theta + \sum 2xy \sin\theta \cos\theta + \sum x^2 \cos^2\theta}{n}} \quad (15)$$

where  $r$  is the radial distance in polar coordinates.

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<sup>18</sup> Much of this argument follows from a derivation by Furfey (1927) and from a general discussion of the BCQ by Salmon (1879).



Introducing Pearson's product moment correlation coefficient ( $\rho$ ):

$$\rho = \frac{\Sigma xy}{n\sigma_x\sigma_y} \text{ or } \Sigma xy = \rho n\sigma_x\sigma_y \text{ and } \Sigma y^2 = n\sigma_y^2 \text{ and } \Sigma x^2 = n\sigma_x^2; \quad (16)$$

$$\text{thus, } r = \sqrt{\sigma_y^2 \sin^2 \theta + 2\rho\sigma_x\sigma_y \sin\theta\cos\theta + \sigma_x^2 \cos^2 \theta}. \quad (17)$$

If  $\rho = 0$ , as in the case of a circular normal distribution, ( $\sigma_x = \sigma_y$ ) then the relationship collapses to:

$$r = \sqrt{\sigma_x^2 (\cos^2 \theta + \sin^2 \theta)} = \sigma_x \quad (18)$$

The circle, of course, is the only figure which satisfies this condition.

Returning to the general case, the relationship may be transformed back to the cartesian system by first squaring both sides (19), and then introducing enough 'r' terms (20) so as to permit utilization of the polar coordinate to cartesian coordinate transformation formulas (21) and (22):

$$r^2 = (\sigma_x^2 \cos^2 \theta + 2\rho\sigma_x\sigma_y \cos\theta\sin\theta + \sigma_y^2 \sin^2 \theta) \quad (19)$$

$$r^4 = (\sigma_x^2 r^2 \cos^2 \theta + 2\rho\sigma_x\sigma_y r^2 \cos\theta \sin\theta + \sigma_y^2 r^2 \sin^2 \theta) \quad (20)$$

$$x = r \cos\theta \quad (21)$$

$$y = r \sin\theta \quad (22)$$

Using the relationship  $(x^2 + y^2) = r^2 (\cos^2 \theta + \sin^2 \theta)$  equation (20) may be expressed in cartesian coordinates as:

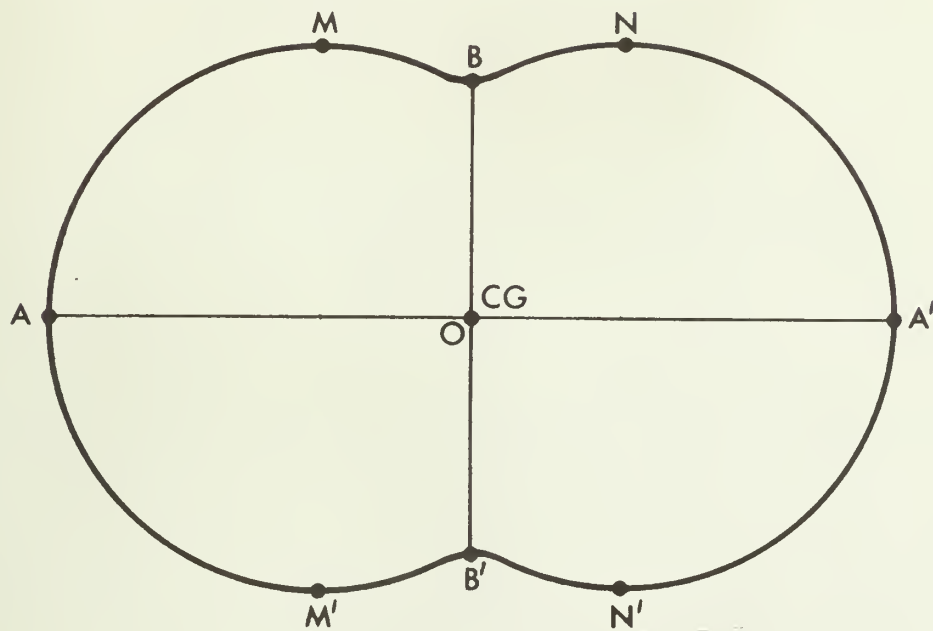
$$(x^2 + y^2)^2 = \sigma_x^2 x^2 + 2\rho\sigma_x\sigma_y xy + \sigma_y^2 y^2. \quad (23)$$

Equation (23) is the standard form of the bicircular quartic.

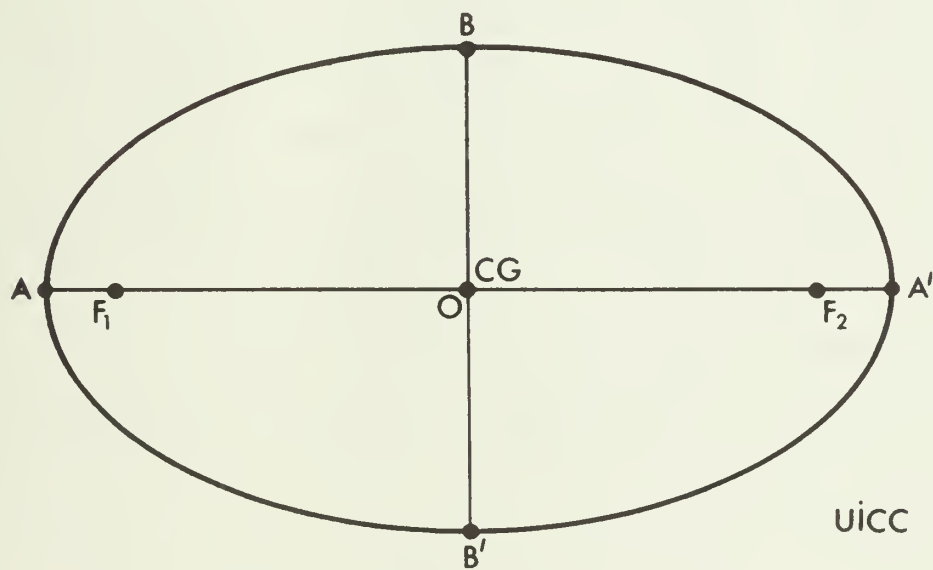
Like the ellipse, the BCQ is a member of the family of curves generated by conic sections. The quartic is produced by three conics. The



Figure 13. Comparison of the Standard Deviation Ellipse and the Bicircular Quartic



Bicircular Quartic with local maxima at  $M$ ,  $N$ , and  $B$  and local minima at  $M'$ ,  $N'$ , and  $B'$ .



UICC

Ellipse with local maximum at  $B$  and local minimum at  $B'$ .



BCQ and SDE have four points in common. These are the intersections of the curves with the major and minor axes (Figure 13, points A, A', B, and B'). Their shapes are clearly different, however, the quartic having three local maxima and minima and the ellipse having only one of each.

In the geographic literature, the bicircular quartic has been described by Caprio (1969; 1970) and Hultquist and others (1970), but they utilized computerized programs which followed dissimilar procedural logic. As a result, they produced two different figures, yet each program made a distinct contribution to centographic research. Caprio iteratively computed the standard deviations along axes rotated by increments of five degrees. This procedure yielded a locus of points sufficient to adequately approximate the BCQ. Hultquist also set out to computerize the BCQ, but utilized an ellipse functional subroutine, thereby producing the SDE and not the BCQ as described in the report. Hultquist's title, "standard ellipse", matches the computerized figure but not the verbal description. This program should thus be used when the SDE is desired.

Geographers have utilized the BCQ to study areal population patterns. Buttner (1972) employed the BCQ to examine social contacts for residents of two planned and two unplanned communities.<sup>19</sup> The BCQ's, called standard ellipses, are shown to be effective figures for contrasting the respective fields of social contacts. Not only is the scope of each field succinctly summarized, but also the directional biases are revealed.

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<sup>19</sup> Buttner (1972) describes in detail the utility of using the ellipse. Reference is made, for instance, to the area of the ellipse, which is based on the major and minor axes. Although the area of the BCQ is often similar, the BCQ is larger. Indeed, in Figure 9 all of the points are on a straight line and thus the ellipse is not shown because its area is equal to zero. The area of the BCQ is  $1/2 \pi \sigma_r$ , where  $\sigma_r$  is the standard radius.



Hyland (1970, p. 78) employed the BCQ to graphically display social interaction.<sup>20</sup> The BCQ's of social trips to friends, organizations, and local relatives identified the spatial scope of each of these interaction fields. The method depicts well the marked contrasts in extent and orientation of the three fields. For example, local relatives appear to be scattered throughout the central city, while friends are confined mainly to the neighborhood.

Brown and Holmes (1971) employed Hultquist's program to analyse intra-urban residential movements. However, they were not dealing with a dot distribution, but rather with a series of migration flows. By transferring all the migration origins to a common point, centrophobic measures may be computed. For example, the BCQ identified the primary directional component as well as the distances traveled.

In summary, since the BCQ and SDE indicate the principal trend in the distribution, they represent improvements over the SR. However, they add only one directional component and rarely is the SDE used to test for bivariate normality. Perhaps their basic advantage lies in providing summary generalizations of complex distributions. Comparing the two, the SDE is a simpler form, but the BCQ is intuitively more satisfying. One would expect the figure which is traced by a multitude of standard deviations along a series of incrementally rotated axes (the BCQ) to be more representative of the original distribution. In reality, with some exceptions,

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<sup>20</sup> Hyland called his figure a "standard deviational ellipse". Clearly, however, he calculated the BCQ: "Employing an arbitrary cartesian grid, the mean center of the points is calculated. A new grid, composed of orthogonal axes in the same scale as the first, is centered on the mean center. The deviation of each point, from the x-axis, is employed to calculate the standard deviation. The axes are then rotated, say by ten degrees, and the procedure repeated. If all the standard deviations along the x-axes are joined, an ellipse will be described." (Hyland 1970, p. 78)



there is little visual difference between the BCQ and the SDE (Figure 13). Still, it must be recognized that the two figures are mathematically distinct, and the appropriate derivation should be used.

Furthermore, the BCQ does not indicate whether the principal deviation is in one or many directions. That distinction can only be made by a measure that is based on directional deviations from a central point.

Sectogram. The figure sensitive to the deviations in all radial directions is the sectogram. It is produced by connecting the locus of points representing the degrees of deviation from the mean center within radial sectors (Figure 14). Schneider (1968), an early proponent of this technique, computed the deviations in eight sectors. He noticed, however, that the within sector deviation was significantly affected by rotating the eight sectors by a few degrees. Thus, depending on the original orientation of the sectors, varying results could occur.

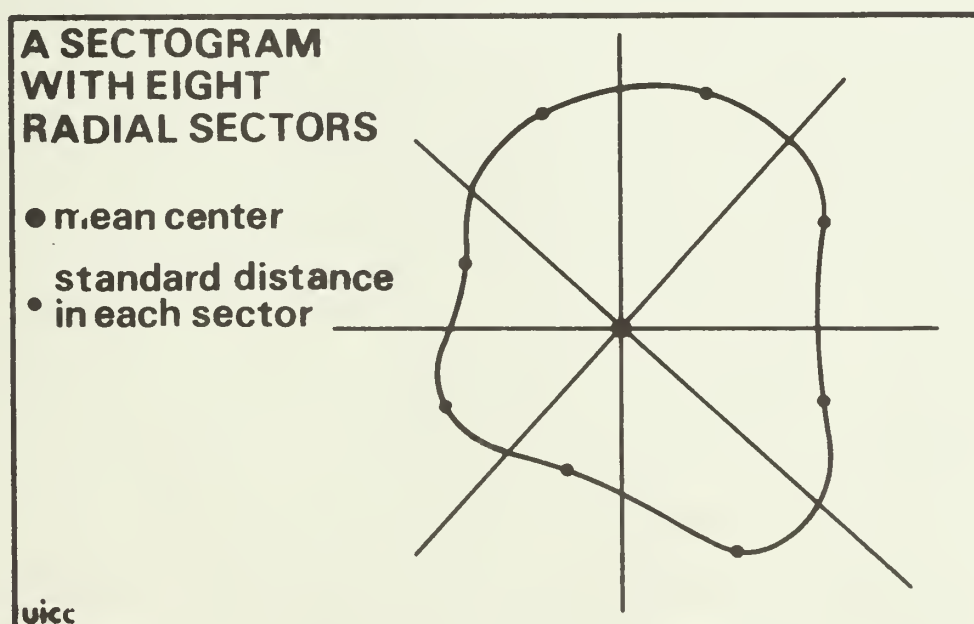


Figure 14



Caprio (1969) modified Schneider's sectogram by decreasing the sector size to twenty degrees. The sectors were rotated by increments of five degrees to produce a new sector overlapping fifteen degrees with the preceeding sector. In this manner seventy-two sectorial deviations were computed and placed on the respective bisectors of each sector, producing the sectogram. Smaller sector sizes and incremental axes rotation both serve to reduce the effects of the original axes orientation in addition to providing a measure with a finer degree of resolution. Caprio also includes a computer program listing of his modified sectogram, used extensively in a study of Newark's population and housing characteristics. The listing is more an explanation of method rather than a program intended for general use.

Unlike the SR, the SDE, and the BCQ, the sectogram does not have a unique solution for a given distribution. The solution is determined in part by the nature of the distribution, but more importantly, it is a function of several free parameters, primarily the axis orientation and the sector size. But basically, it is purely a descriptive figure which is responsive to unusual distributional shapes. In contrast to the BCQ and SDE it is a more effective geometric form when more detail is desired. But this property contributes to its major disadvantage: a lack of characteristics which permit comparisons of sectograms. For example, there is no such thing as a radius or major axis of a sectogram. Certainly, only one's imagination limits the number of indices which may be computed, such as the range of the sector deviations, the area of the sectogram and the autocorrelation of sector deviations. These indices describe the properties of the sectogram, but few are as obvious as the major and minor axes of the ellipse, and none have been universally accepted.



## SUMMARY AND EVALUATION

It is evident at this point that each centographic measure characterizes a particular property of a distribution. The mean center has a distinctly different derivation than the PMAT and, with the exception of the circular normal distribution, is not located at the same point. The mean center is pulled in the direction of the most distant points, while the PMAT is influenced by the clustering of points. Therefore, the mean center is biased by the extreme distance values of those at the periphery of the study area, whereas the PMAT is not.

The shape of the distribution is also significant when a phenomena is studied through time. If the objective is, for example, to illustrate the historical migration of people to Chicago, then the PMAT would hardly be of any value. The PMAT has moved very little, being in the Chicago area for the last fifty years, belying the actual population shifts that have occurred. Given a different population scatter, the PMAT may have represented extremely well the temporal shifts.

In summary, the measure of central tendency to be utilized is dependent upon (1) the objective of the study, and (2) the nature of the distribution. In most instances, the choice is between the mean center and the PMAT. The orthogonal median point can be used as a quick approximation of the other two measures with the understanding that its accuracy is uncertain. Unlike the other measures, the orthogonal median point can be quickly determined without converting the points to coordinate values.

The conclusions regarding the measures of dispersion are also general, but some specific observations can be made. One characteristic of the standard radius that distinguishes it from the other three measures--the SDE, the BCQ, and the sectogram--is that it is primarily used for



analytical purposes. As descriptive measures all have distinct disadvantages. Since considerable detail is lost, the SR, the SDE, and the BCQ are not good descriptive figures. Each has traits which permit comparisons among similar figures. To the contrary, comparisons among sectograms prove to be awkward. But as a detailed descriptive form, the sectogram is superior. Yet, if only a simple geometric figure is desired then the standard deviational ellipse is appropriate.

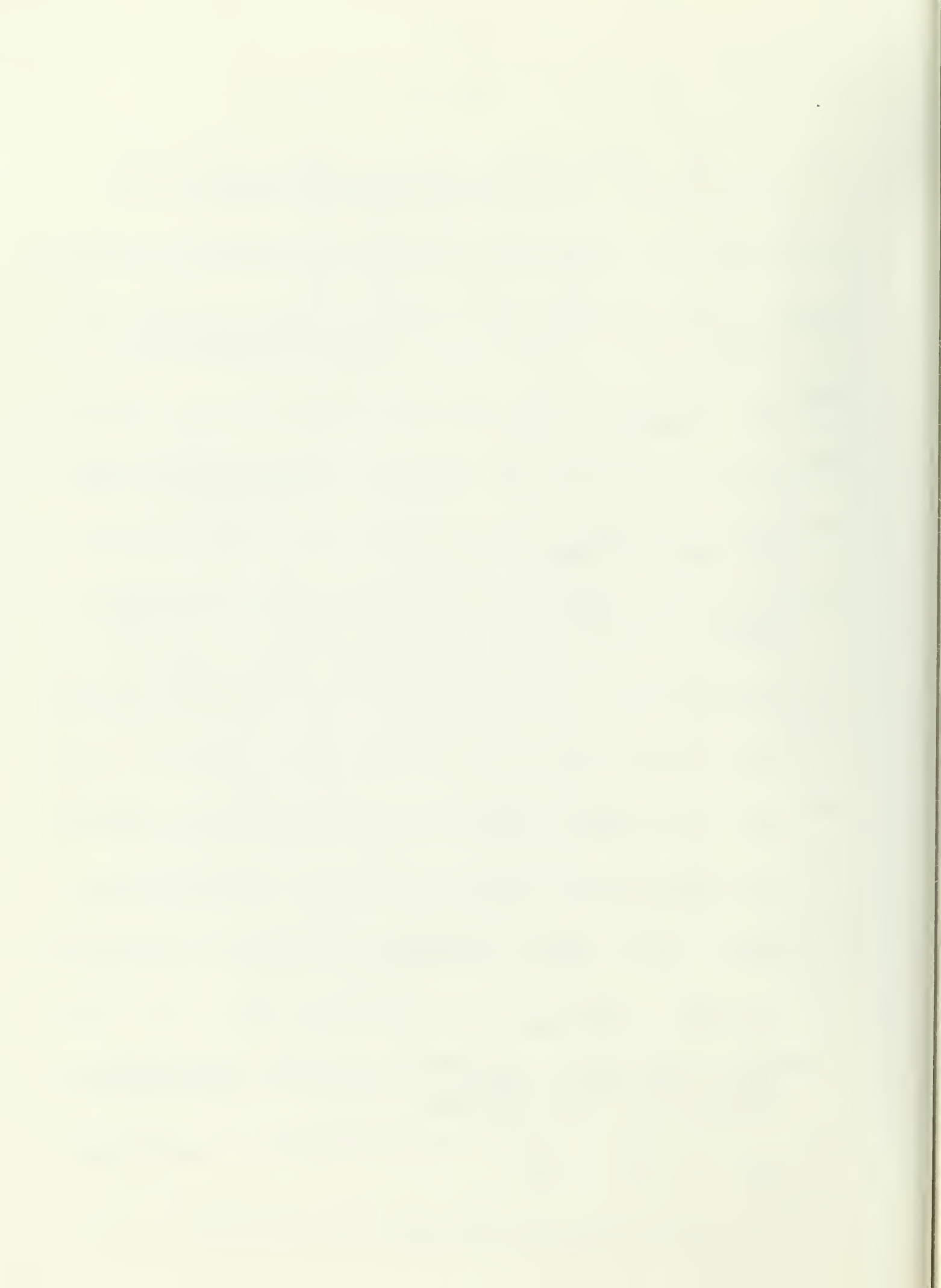
We may conclude that of the four measures, the SR is analytic, the sectogram is purely descriptive, and while the SDE may be used as an analytic measure it is predominantly used for descriptive purposes. The BCQ is an adaptation of the SDE, and represents no improvement over the SDE, and perhaps leads only to confusion. Geometrically, their shapes are similar and only diverge in cases where dispersion is principally in one dimension. In such cases, the BCQ is clearly misleading and should not be employed (Figure 9). Its use stems from an intuitive appeal. It would seem that connecting many standard deviations is better than connecting only four as in the SDE. This proves not to be true. For these reasons, the use of the BCQ should be abandoned, leaving us wuth three basic choices--the analytic SR, the "simple descriptive" SDE, and the "detailed descriptive" sectogram.

Judgment must be exercised in the selection of centrophraphic measures. Their limitations must be heeded, but in general their use is encouraged. Few methods can convey as much information as a few measures of centrophraphy. Although detail may be sacrificed, centrophraphic measures are valuable geographic generalizations worthy of wide-spread use.



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